Pergamon

J. Appl. Maths Mechs, Vol. 60, No. 2, pp. 343–347, 1996

0021–8928/96 \$24.00+0.00

S0021–8928(96)00045-7

A JUSTIFICATION FOR INVESTIGATING THE DYNAMIC PROPERTIES OF AN ELASTIC BAR USING A MODEL OF A SYSTEM OF COUPLED RIGID BODIES†

I. A. BOLGRABSKAYA

Donetsk

(*Received* 23 *May* 1995)

The convergence of the solutions of the equations of a finite-dimensional model of the oscillations of a beam (n masses coupled by elastic hinges) to the solutions of a system with distributed parameters as $n \to \infty$ is proved.

A constructive algorithm for finding the free and resonance vibrations of the finite dimensional model being considered has been presented earlier [1]. A review of the results on the use and study of a system of rigid bodies is given in [2].

1. AN ELASTIC ROD AND A SYSTEM OF COUPLED RIGID BODIES

Consider a rod which has a constant cross-section and an unchanging stiffness (a homogeneous rod). We assume that the elements of the rod undergo only translational motion during the vibrations, that both ends of the rod are free and that there are no external forces or moments. The equations for small vibrations of the elastic rod and the boundary conditions then have the form

$$EI\partial^4 y/\partial x^4 + S\rho\partial^2 y/\partial t^2 = 0 \tag{1.1}$$

$$x = 0, \ \ x = l: \ \ EI\partial^2 y/\partial x^2 = 0, \ \ EI\partial^3 y/\partial x^3 = 0 \tag{1.2}$$

where EI is the stiffness of the rod, l is its length, S is the cross-section area, ρ is the density and the OX axis is directed along the undeformed axis of the rod.

We now consider the motion of a system of coupled rigid bodies (SCRB) which simulates the small vibrations of the elastic rod in the formulation described above. In the finite-dimensional case, it can be represented by $N-1$ point masses with masses m arranged along an elastic line which is simulated by N weightless rods S_k coupled by cylindrical hinges. It is assumed that the motion occurs in the OXY plane.

We introduce the following systems of coordinates:

$OXYZ$ (unit vectors $\mathbf{e}_x$, $\mathbf{e}_y$, $\mathbf{e}_z$) is a fixed system of coordinates, the OX axis of which at the initial instant $t = t_0$ is directed along the axis of the body S_1;

O_0XYZ is a system associated with the point O_0 which belongs to the body S_1, the axes of which are collinear with the corresponding axes of the fixed system;

$O_0X'Y'$ (unit vectors $\mathbf{i}$ and $\mathbf{j}$) is a system in which $O_0X' \parallel O_0O_N$, where O_N is a point on the axis of the body S_N;

$O_{k-1}X_kY_k$ is a system associated with the body S_k and, in it, $O_{k-1}X_k \| O_{k-1}O_k$ ($k = \overline{1, N}$).

We will define the position of the system $O_0X'Y'$ with respect to the fixed angle ψ and the position of the coupled system with respect to $O_0X'Y'$ by the angle ψ_2 ($k = \overline{1, N}$).

As in [4], during the motion of the free system [1], it is possible to separate out its motion as a whole (the large motion) and the relative elastic vibrations (the small motion). Here, the large motion is determined by the change in the coordinates of the point $O_0(x_0, y_0)$ and the angle ψ, while the small motion is determined by the change in the angle ψ_k. Since only small deformations of the rod are considered in the continuous formulation then, also in the SCRB, we shall study the case when the angles ψ_k are small. The kinetic energy and potential energy of the SCRB under consideration are

$$T = \frac{m}{2}\sum_{k=0}^{N} v_k^2, \quad \Pi = \frac{1}{2}\varkappa^2\left[\sum_{k=2}^{N}(\psi_k - \psi_{k-1})^2 + o(\psi_k^2)\right] \tag{1.3}$$

†*Prikl. Mat. Mekh.* Vol. 60, No. 2, pp. 346–350, 1996.

where $\varkappa^2$ is the stiffness of an elastic hinge, $v_k = v_k(x_0, y_0, \psi, \psi_k)$ is the velocity of the pole O_k ($k = \overline{0,N}$) and $o(\psi_k^2)$ are terms of greater than the second order of smallness in ψ_k.

We conclude from (1.3) that the potential energy is positive definite with respect to some of the variables and, in fact, with respect to ψ_k and, since the kinetic energy is a positive definite function it follows from the results previously obtained [5] that, provided the ψ_k are small at $t = t_0$, they remain small during the whole duration of the motion as a consequence of the stability of the motion of the system with respect to these variables. We will denote the coordinates of the points O_k ($k = \overline{0,N}$) in the system $O_0X'Y'$ by u_k, w_k. Then,† for small deflections, we assume in the linear approximation that

$$\psi_k = (w_k - w_{k-1})/h, \quad u_k = kh \quad (k = \overline{1,N}) \qquad (1.4)$$
$$u_0 = w_0 = w_N = 0, \quad u_N = Nh, \quad h = l/N$$

where h is the distance between O_k and O_{k+1} ($k = \overline{0,N}$).

The velocity of the point O_k is equal to $\mathbf{v}_k = \dot{\mathbf{OO}}_k = \mathbf{v}_0 + \dot{\mathbf{O}_0\mathbf{O}}_k$ and can be expressed as

$$\mathbf{v}_k = \mathbf{v}_0 + (\dot{w}_k + kh\dot{\psi})\mathbf{j} - w_k\dot{\psi}\mathbf{i} \qquad (1.5)$$
$$\mathbf{v}_0 = \dot{x}_0\mathbf{e}_x + \dot{y}_0\mathbf{e}_y$$

We substitute expressions (1.4) and (1.5) into (1.3). The kinetic energy and potential energy, apart from terms of the second order of smallness, are as follows:

$$T = \frac{1}{2}Nm(\dot{u}_p^2 + \dot{w}_p^2) + \frac{m}{2}\sum_{k=1}^{N}[-2\dot{\psi}\dot{u}_p w_k + w_k^2\dot{\psi}^2 + 2\dot{w}_p(\dot{w}_k + kh\dot{\psi}) + \dot{w}_k^2 + k^2h^2\dot{\psi}^2 + 2kh\dot{\psi}\dot{w}_k]$$
$$\Pi = \frac{\varkappa^2}{2h^2}\left[(w_2 - 2w_1)^2 + \sum_{k=3}^{N-1}(w_{k-1} - 2w_k + w_{k+1})^2 + (w_{N-2} - 2w_{N-1})^2\right]$$

Here, $\dot{u}_p = \dot{x}_0\cos\psi + \dot{y}_0\sin\psi$, $\dot{w}_p = -\dot{x}_0\sin\psi + \dot{y}_0\cos\psi$

The system has three integrals

$$M\dot{u}_p - m\dot{\psi}\sum_{k=1}^{N-1}w_k = C_1$$
$$M\dot{w}_p + ma\dot{\psi} + m\sum_{k=1}^{N-1}\dot{w}_k = C_2 \qquad (1.6)$$
$$\dot{\psi}\left(a_1 + \sum_{k=1}^{N-1}w_k^2\right) + a\dot{w}_0 - \dot{u}_p\sum_{k=1}^{N-1}w_k + h\sum_{k=1}^{N-1}k\dot{w}_k = C_3$$
$$(M = Nm,\ a_1 = h^2N(N-1)(2N-1)/6,\ a = hN(N+1)/2)$$

We will now consider the case when, at the initial instant, the momentum and the angular momentum are equal to zero. This means that the constants of integration are equal to zero. In particular, we can then assume that, at the initial instant, system (1.6) has a null solution. Assuming that the variables ψ, u_p, w_p are small when $t = t_0$, we consider them to be small during the whole time of the motion. In this case, we denote the coordinates of the points O_k ($k = \overline{0,N}$) in the fixed basis by x_k, y_k, assume that y_k are small and obtain the following equations of the small vibrations (see the paper cited in the footnote)

$$\ddot{y}_0 + \varkappa_1^2(y_2 - 2y_1 + y_0) = 0$$
$$\ddot{y}_1 + \varkappa_1^2(y_3 - 4y_2 + 5y_1 - 2y_0) = 0$$
$$\ddot{y}_k + \varkappa_1^2(y_{k-2} - 4y_{k-1} + 6y_k - 4y_{k+1} + y_{k+2}) = 0 \quad k = \overline{2, N-2} \qquad (1.7)$$
$$\ddot{y}_{N-1} + \varkappa_1^2(y_{N-3} - 4y_{N-2} + 5y_{N-1} - 2y_N) = 0$$
$$\ddot{y}_N + \varkappa_1^2(y_{N-2} - 2y_{N-1} + y_N) = 0, \quad \varkappa_1^2 = \varkappa^2/(mh^2)$$

†BOLGRABSKAYA I. A., Equations of systems of coupled rigid bodies and of small vibrations of elastic roads. In Dynamics of systems of coupled rigid bodies and bodies with cavities containing a liquid. Preprint No. 90-03. Inst. Prikl. Mat. Mekh., Akad. Nauk UkrSSR, Donetsk, 1990.

2. COMPARISON OF THE SOLUTIONS FOR THE ROD AND FINITE-DIMENSIONAL MODELS

It is easily seen that the equations of motion (1.7) can be treated as a finite-difference approximation of Eqs (1.1) and (1.2) with respect to the spatial variable $x_k = kh$. This fact subsequently enables us to use theorems which have been proved in the theory of finite difference schemes to prove that the solution of the finite-dimensional problem (SCRB) converges to the solution of the continuous problem (an elastic rod) which corresponds to it as regards its formulation. According to the results obtained earlier [6], the solution of the finite-dimensional problem converges to the solution of the continuous problem when two conditions are satisfied: the equations of the finite-dimensional model must approximate the equations of the continuous model and, furthermore, the finite-difference scheme, to which the system of ordinary differential equations corresponds in the case under consideration, must be stable.

Let L be a continuous differential operator and let L_h be a difference operator. Then, Eqs (1.1) and (1.2) and system (1.7) can be correspondingly written as

$$Ly = 0 \tag{2.1}$$

$$L_h y_h = 0 \tag{2.2}$$

Then, if $y(x_k)$ is the solution of the continuous problem (2.1) when $x = x_k$, we have $L(y(x_k)) = 0$, $L_h(y(x_k)) = \partial f_h$.

By definition [6], the solution of the finite-difference system (2.2) approximates the solution Y of the continuous system (2.1) in the case when $\| \delta fh \| \to 0$ when $h \to 0$.

In (1.7), let us put

$$\varkappa^2 = EI/h, \quad m = \rho Sh \tag{2.3}$$

It has been proved (see the paper cited in the footnote) that, subject to condition (2.3), the equations of motion of the SCRB (1.7) converges to the equations for small vibrations of an elastic rod (1.1), (1.2). It follows from this that the equations of motion of the SCRB approximate to the equations for small vibrations of elastic rods.

Since the approximation has already been established, we shall concentrate on proving the stability. According to the definition in [7], the difference problem (2.2) being considered is stable if, for any $f_h(t) \in F_h$, $\| f_h \|_{Fh} < \delta$ the equation $L_h y_h = f_h$ has a unique solution when $\| y_h \| \leqslant C\delta$, where C is independent of h and $\| \cdot \|$ is any of the norms [8] introduced in the space of the functions Y.

The integrals (1.6) with null integration constants have the following form in the new variables

$$\sum_{k=0}^{N} \dot{y}_k = 0, \quad \sum_{k=1}^{N} k\dot{y}_k = 0 \tag{2.4}$$

We now make the change of variables

$$z_k = y_k - y_{k-1} \quad (k = \overline{1, N}), \quad \upsilon_k = z_k - z_{k+1} \quad (k = \overline{1, N-1}) \tag{2.5}$$

in (1.7) and we then obtain the following system

$$\ddot{V} + GN^4 AV = 0; \quad G = EI/(\rho S l^4), \quad N = l/h \tag{2.6}$$

where G is a constant which is independent of h, A is a symmetric pentadiagonal matrix of the nth order ($n = N - 1$) with elements a_{ij} defined by the equalities

$$a_{ij} = \begin{cases} 6, & j = i, \\ -4, & j = i+1, \\ 1, & j = i+2 \\ 0, & j > i+2 \end{cases} \quad i \leqslant j < n$$

and the elements below the principal diagonal are obtained by symmetric reflection $V = (\upsilon_1, \ldots, \upsilon_n)^T$. System (2.4), (2.6) is equivalent to (1.7). After finding υ_k ($k = \overline{1, n}$) from (2.6), we find y_0 and $z_1 = y_1 - y_0$ from (2.4) and the remaining y_k ($k = \overline{2, n}$) from (2.5).

Hence, to estimate the solution of system (1.7) and to prove stability, it is necessary to estimate the solution of system (2.6) and to show that small perturbations of its right-hand side lead to small perturbations of the solution which are independent of h. The establishment of this fact is also evidence of the stability of system (2.6) and, together with it when account is taken of (2.4) and (2.5), of system (1.7).

So, we shall consider the system

$$\ddot{V}+GN^4AV=f(t),\quad \|f(t)\|<\delta$$

Since A is a real symmetric matrix, an orthogonal matrix B always exists which will reduce it to diagonal form. We obtain

$$\ddot{U}+\Lambda U=B^{-1}f(t) \tag{2.7}$$

$$U=B^{-1}V,\quad \Lambda=\mathrm{diag}(\lambda_1,\ldots,\lambda_n)$$

where $\lambda=(\lambda_1,\ldots,\lambda_n)$ is the vector of the eigenvalues of the matrix $C=\{c_{ij}\}=\{GN^4a_{ij}\}$.

The general solution of Eq. (2.7) is

$$U=U_0\cos\sqrt{\lambda}t+\frac{1}{\sqrt{\lambda}}\dot{U}_0\sin\sqrt{\lambda}t+\frac{1}{\sqrt{\lambda}}\int_0^T B^{-1}f(\tau)\sin\sqrt{\lambda}(t-\tau)d\tau \tag{2.8}$$

Let us assume that $\|V_0\|\leqslant\varphi_0$, $\|\dot{V}_0\|\leqslant\psi_0$ at the initial instant, where $\|\cdot\|$ is the Euclidean norm of a vector. Since B is the orthogonal transformation $\|U\|=\|B^{-1}V\|=\|V\|$, on estimating the norm of (2.8), we arrive at the inequality

$$\|V\|=\|U\|\leqslant\varphi_0+(\psi_0+\delta T)/\min_k|\sqrt{\lambda_k}| \tag{2.9}$$

Thus, to prove stability, it is necessary to show that a constant d exists which is independent of h such that $1/\min_k|\lambda_k|<d$, where λ_k are the eigenvalues of the matrix C.

We will denote the eigenvalues of the matrix M by C. Then, on taking account of (2.7), we have [9]

$$\lambda(C)=GN^4\lambda(A)=G(n+1)^4\lambda(A)$$

All of the angular minors of the matrix A can be calculated in explicit form

$$\Delta_i=(i+1)(i+2)^2(i+3)/12>0\quad (i=\overline{1,n})$$

and it follows from Silvester's criterion that the real matrix A is positive definite and all its eigenvalues are real and positive [8, 9]. Let A^i be a matrix of the ith order ($i\leqslant n$) obtained from the first i rows and i columns of the matrix A. We denote the eigenvalues by $\lambda(A^i)=(\mu_1^i,\ldots,\mu_i^i)$, assuming that they are numbered in increasing order of magnitude. Then

$$\min_k(\lambda_k|=GN^4\mu_1^n \tag{2.10}$$

We now consider the matrix B^i in which $b_{11}=b_{ii}=5$ and the remaining $b_{ij}=a_{ij}$. It is obvious that the equality

$$B^i=(D^i)^2\quad i\geqslant 2 \tag{2.11}$$

is satisfied, where D^i is a symmetric matrix of the ith order, the elements of which are

$$d_{km}^i=\begin{cases}2, & m=k,\\ -1, & m=k+1,\quad k<m\leqslant i\\ 0, & m>k+1;\end{cases}$$

The eigenvalues of the matrix D^i are known [7]. It then follows from (2.11) [9] that

$$\lambda_k(B^i)=\lambda_k^2(D^i)=16\sin^4\frac{k\pi}{2(i+1)}\quad (k=\overline{1,i}) \tag{2.12}$$

We shall use the notation $\lambda_k(B^i)=\nu_k^i$. Then, from (2.12), we obtain

$$\nu_k^i<\nu_l^i,\quad k<l\leqslant i;\quad \nu_k^i>\nu_k^s,\quad i<s\leqslant n \tag{2.13}$$

Since the eigenvalues of the matrix are the roots of its characteristic polynomial, μ_k^s, ν_k^s are the roots of the equations

$$\Delta_k(\lambda)=A^k-\lambda E^k,\quad \delta_k(\lambda)=B^k-\lambda E^k$$

respectively, where E^k is the identity matrix of the kth order ($k \leqslant n$).

We now represent Δ_k in the following manner

$$\Delta_k(\lambda) = \Delta_{k-1}(\lambda) + \sum_{i=1}^{k} \delta_i(\lambda), \quad k = \overline{2,n} \tag{2.14}$$

$$\Delta_1 = = -\lambda + 6$$

Next, by taking account of the fact that all the roots of the matrices A^k, B^k are real, Eq. (2.14) can be written as

$$\Delta_k(\lambda) = \prod_{i=1}^{k} (-\lambda + \mu_i^k) = \prod_{i=1}^{k-1} (-\lambda + \mu_i^{k-1}) + \sum_{i=1}^{k} \prod_{l=1}^{i} (-\lambda + \nu_l^i), \quad k = \overline{2,n} \tag{2.15}$$

When $k = 1$, we have $\mu_1^1 = 6$, $\nu_1^1 = 5$.

On considering (2.15) for $k = 2, \ldots, n$, we successively find, when conditions (2.13) are satisfied, that $\Delta_k(\lambda) > 0$ when $\lambda > \nu_1^k$, and it follows from this that

$$\mu_1^k > \nu_1^k, \quad k = \overline{2,n} \tag{2.16}$$

Finally, using (2.12), we have from (2.10) and (2.16) that

$$\left(\min_k |\lambda_k|\right)^{-1} < \left[16G(n+1)^4 \sin^4 \frac{\pi}{2(n+1)}\right]^{-1} \tag{2.17}$$

Using the relation

$$\lim_{n\to\infty} 16(n+1)^4 \sin^4 \frac{\pi}{2(n+1)} = \pi$$

we conclude that, for sufficiently large n (or small h), it follows from (2.17) that $(\min_k |\lambda_k|)^{-1} < d$, where the constant d is independent of h.

So, an estimate of the smallest eigenvalue $\lambda(C)$ which is independent of h has been obtained which enables us to estimate the solution of (2.8), taking account of (2.9), as follows:

$$\|V\| \leqslant \varphi_0 + (\psi_0 + \delta T)/\sqrt{d} = C\delta \tag{2.18}$$

Relation (2.1) gives an estimate of the solution of system (2.6) which is independent of h as a consequence of which it follows from (2.4) and (2.5) that the solution of system (1.7), in the case of small perturbations of the right-hand side, also changes by an amount which is independent of h, that is, system (1.7) is stable.

Next, by taking account of the fact that system (1.7) approximates to (1.1), (1.2), we conclude that its solution converges to the solution of the continuous system.

I wish to thank A. Ya. Savchenko for his interest and for useful discussions.

REFERENCES

1. SAVCHENKO A. Ya., BOLGRABSKAYA I. A. and KONONYKHIN G. A., *Stability of the Motion of Systems of Coupled Rigid Bodies*. Naukova Dumka, Kiev, 1991.
2. BOLGRABSKAYA I. A. and SAVCHENKO A. Ya., *Stability of Steady Motions of Systems of Coupled Rigid Bodies*. Mekh. Tverd. Tela, Naukova Dumka, Kiev, 21, 62–73, 1989.
3. TIMOSHENKO S. P., *Vibrations in Engineering*. Nauka, Moscow, 1967.
4. COCHRAN J. E., BATSON J. L. and CHRISTENSEN D. E., Dynamic coupling of gross and fine motions of a flexible spinning free rocket. *AIAA: Atmos. Flight Mech. Conference*, Hollywood, CA, pp. 182–194, 1977.
5. RUMYANTSEV V. V. and OZIRANER A. S., *Stability and Stabilization of Motion with Respect to Some of the Variables*. Nauka, Moscow, 1987.
6. RYABEN'KII V. S. and FILIPPOV A. F., *The Stability of Difference Equations*. Gostekhizdat, Moscow, 1956.
7. GODUNOV S. K. and RYABEN'KII V. S., *Difference Schemes*. Nauka, Moscow, 1973.
8. GANTMAKHER F. R., *Matrix Theory*. Nauka, Moscow, 1967.
9. MARCUS M. and MINC H., *A Survey of Matrix Theory and Matrix Inequalities*. Allyn & Bacon, Boston, MA, 1964.

Translated by E.L.S.